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Twisted surfaces

Victor A. Nicholson
Department of Mathematical Sciences, Kent State University, Kent OH 44240, USA
E-mail: vnichols@math.kent.edu

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This article proves the laundry embedding theorem. It considers surface triples (S, G, J) in S^3 where S is a 2-manifold with boundary, G is a circle-with-chords, and J is an arc. The surfaces satisfy an embedding condition called laundry which is similar to being unknotted. The theorem gives elementary necessary and sufficient conditions for two triples to be equivalent by ambient isotopy. The theorem introduces a new topological invariant called turn. The surfaces of interest can arise from the augmented ribbon model of unknotted single domain protein.

KEY WORDS: surface, linking, protein

AMS subject classification: 57M25, 57M15, 92C40

1. Introduction

Modeling protein as it moves in space requires that the backbone be preserved throughout the movement. The backbone is represented by an oriented arc in the augmented ribbon model [1]. The requirement that a homeomorphism preserve this oriented arc makes turn (defined below) a topological property. An obvious mathematical requirement is to connect the ends of the arc to make a circle. The chords are suggested by bonds that stabilize the backbone's position. Proteins denature or unfold. This property suggested the laundry condition.

All spaces are assumed to be piecewise linear. Consider a graph G consisting of a circle subdivided by a set of points $\{a_0, \ldots, a_n, b_0, \ldots, b_n\}$ together with a set of n oriented *chords* $E_i = a_i b_i, i = 1, \ldots, n$. The points are labeled so that a_0 and b_0 are adjacent on the circle, the oriented *arc* $J = a_0 b_0$ in the circle contains all of the vertices, $a_0 < a_1 < \cdots < a_n$, and $a_i < b_i$, for $i = 0, \ldots, n$. Let $E_0 = a_0 b_0$ (also called a chord) be the closure of the complement of J in the circle. A *circle-with-chords* is a pair (G, J). Two graphs (G, J) and (G', J') are said to be the *same graph* if there is a homeomorphism $h : (G, J) \to (G', J')$ that preserves the orientations of the arcs J and J'.

Suppose G is in R^3, $p : R^3 \to R^2$ is the orthogonal projection, and $P_i = p(E_i), i = 0, \ldots, n$, are the projected chords. The graph (G, J) is said to be in *laundry position* if (1) every vertical line intersects G in 0, 1 or 2 points and, if

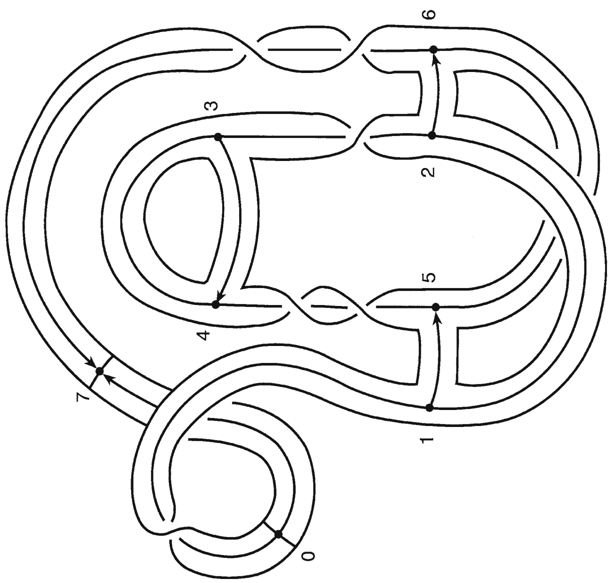

Figure 1. A laundry surface in three-dimensional space.

2, neither of them is a vertex of G, (2) the arc J of G lies in the x-axis, (3) each P_i is an arc in the bottom half of the xy-plane, (4) any pair of P_i intersect in at most one point, (5) no P_i meets the arc of G at an interior point of P_i, and (6) if P_i meets P_j and P_k with $i < j < k$, then P_i meets P_j first and then P_k. The central feature of laundry position is condition (6) which requires that the order in which a chord meets other chords is the same as their order along the arc. Simple examples of graphs that are isotopic to graphs in laundry position are the face of a tennis racquet or a Hamiltonian graph in the plane whose vertices all have degree three. Note that the arc J defines the "laundry line".

Throughout this article a *surface* will refer to a triple (S, G, J) consisting of a 2-manifold with boundary S, a circle-with-chords G, and its arc J satisfying the condition that S is a regular neighborhood of G in S. This allows the surface to freely twist around the edges in the graph. Two surfaces (S, G, J) and (S', G', J') are said to be *equivalent* if there is an orientation preserving homeomorphism h of S^3 onto itself such that $h(S, G, J) = (S', G', J')$ and h preserves the orientations of the arcs J and J'. That is, equivalence is ambient isotopy. A *laundry* surface (S, G, J) in S^3 is one that is equivalent to a surface (S', G', J') with (G', J') in laundry position.

A laundry surface is illustrated in figure 1. There is an ambient isotopy that moves its graph into laundry position as shown in figure 2. The twists (half-twists) in the resulting surface are given by the numbers associated with the edges.

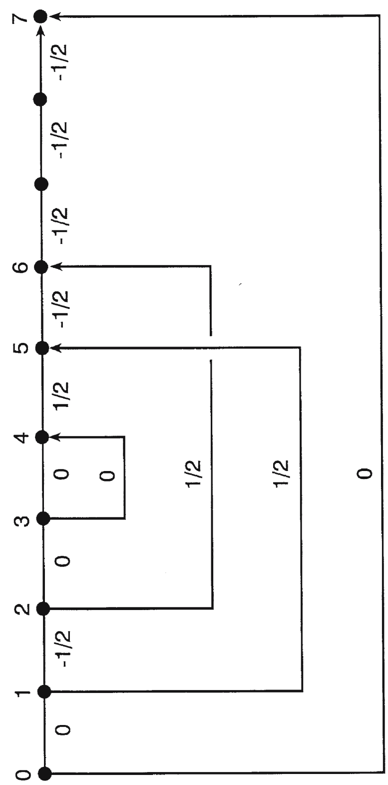

Figure 2. The graph in figure 1 is isotopic to this graph in laundry position. The numbers denote twists in the surface resulting from the isotopy.

For $i = 0, \ldots, n$, let $I_i = a_i b_i$ a subarc of J. Let X denote the family of cycles $X_i = E_i \cup I_i, i = 0, \ldots, n$, oriented in the direction of the chord E_i. The cycles in X are a basis for the first homology group of G and S. The *linking matrix* is $M = (lk(X_i' \, X_j))$, where X_i' is X_i pushed off in "both directions". If S is orientable and A is the Seifert matrix then $M = A + A^T$. The linking matrix represents the Gordon and Litherland form [2] discussed in [3]. Note that the oriented arc J determines the cycle basis and the order of the rows and columns in the matrix.

The linking matrix is not sufficient to determine equivalence. For example, let (G, J) have only two chords, E_1 and E_2, so that the intervals I_1 and I_2 do not intersect. Let (G_1, J_1) and (G_2, J_2) be embeddings of G in R^2 such that G_1 has both chords in the interior and G_2 has one in the interior and one in the exterior of the cycle X_0 (the circle). Let S_1 and S_2 be regular neighborhoods of G_1 and G_2 in R^2, respectively. The pairs (S_1, G_1) and (S_2, G_2) are ambient isotopic but J_1 cannot be carried to J_2.

Suppose e is an edge in J both of whose vertices have degree three. Such as the edge $e = b_1 a_2$ in the preceding examples. Suppose D is a small regular neighborhood of e in S. Place the label "a" on both points in $\partial D \cap J$ and the label "b" on both points where ∂D meets the chords. The cyclic order of these points in ∂D is either $(aabb)$ or $(abab)$. Define the *turn* at e to be 0 if the order is $(aabb)$ and 1, otherwise. The turn at an edge is an invariant of ambient isotopy for surfaces. It isn't necessary to know the turn at every edge. The arcs I_i and I_j are said to overlap if $I_i \cap I_j$ is nonempty, I_i is not a subset of I_j, and I_j is not a subset of I_i. Each pair of overlapping arcs with $1 \leqslant \iota < j \leqslant n$ defines an ordered pair (i, j) called an *overlapping pair*. The *overlap graph O* for G has vertex set $\{1, \ldots, n\}$ and edge set $\{ij \mid (i, j)$ is an overlapping pair$\}$. The set of *interior first-edges* of G is $\{uv \subset J \mid v = a_i$ for $i = \min\{j \mid j \in C\}$, *for each component C of O not containing* $1\}$. The overlap graph for the above examples has two vertices and no edges making the edge $e = b_1a_2$ an interior first-edge. Two surfaces (S, G, J) and (S', G', J') are said to have the *same turns* if they have the same turn at each interior first-edge.

The laundry embedding theorem says that the equivalence of two laundry embeddings of the same protein model in space (or computer simulation) can be determined by checking only the linking matrix and turns.

Theorem 1 (Laundry embedding). Two laundry surfaces are equivalent iff they have the same graph, linking matrix, and turns.

The *doubled-delta move* [4] is illustrated in figure 3. The move preserves the graph, linking matrix, and turns of a surface. The laundry embedding theorem implies that a doubled-delta move must result in either an equivalent surface or one that is not a laundry surface. The strips in figure 3 can be arbitrarily numbered and a direction arbitrarily chosen for the strip labeled 1. Note that the order in which strip 1 meets the other two changes. If one figure satisfies condition (6) for laundry position then the other may not.

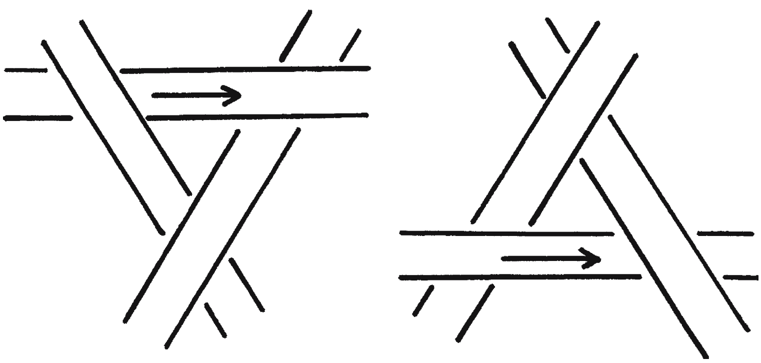

Figure 3. The doubled-delta move.

2. Lemmas

The proof of theorem 1 begins with a series of lemmas. Let $I = [-1, 1]$. Let $I^2 = I \times I$ in R^2 and let D_1 be the disk of radius 1/2 in I^2. The *cube* is $I^3 = I^2 \times I$, its *ends* is $E = I^2 \times \{-1, 1\}$, its *cylinder* is $W^3 = D_1 \times I$, and its *center arc* is $A = \{(0, 0)\} \times I$. Let R_π denote the set of homeomorphisms of the cube consisting of the identity and the three rotations of π radians about each of the coordinate axes. Define strips having one of three types, L, U, or R corresponding to twists in I^2 of $-1/2$, 0, or 1/2, respectively. The cylinder W^3 contains a strip F of each type such that $\partial F \subset \partial W^3$ and the strip is carried to itself by each of the rotations in R_π.

Lemma 1. Suppose F is a strip of any type in the cube I^3 and A is the center arc. Suppose $f : I^3 \to I^3$ is an orientation preserving homeomorphism such that $f(F) = F$ and $f(A) = A$. Then the map f is isotopic to a rotation r in R_π by an isotopy of I^3 that leaves $F \cap \partial I^3$ and $A \cap \partial I^3$ setwise fixed.

Let A_1 and A_2 denote the two components of $F \cap (\partial D_1 \times I)$. The homeomorphism f carries the arc A_1 to either itself or the arc A_2 and preserves or reverses the orientation. There is a rotation r in R_π such that $rf(A_1) = A_1$ preserving its orientation. By the homotopy extension theorem [5, p. 180], there is an isotopy α such that $\alpha_1 rf$ is the identity on F and α leaves $F \cap \partial I^3$ and $A \cap \partial I^3$ setwise fixed. Let E denote the ends of I^3 and $K = F \cap E$. Since E and $\alpha_1 rf(E)$ are regular neighborhoods of K in ∂I^3, it follows from the uniqueness of regular neighborhoods [6, p. 38] that there is an isotopy β of ∂I^3 fixed on K such that, $\beta_1\alpha_1 rf(E) = E$. There is an isotopy γ of ∂I^3 leaving K fixed such that $\gamma_1\beta_1\alpha_1 rf$ is the identity on E. There is a pinched collar U of ∂I^3 pinched at K missing $F - K$ [5, p. 41]. The isotopy $\gamma\beta$ extends to I^3 with support in U [6, p. 37]. Thus $\gamma_1\beta_1\alpha_1 rf$ is a homeomorphism of the cube which leaves the ends fixed. So it can be considered as a restriction of an orientation preserving homeomorphism of a solid torus. That is, isotopic to a Dehn twist [7]. Since it also keeps F fixed it must be the trivial Dehn twist. There is an isotopy δ such that $\delta_1\gamma_1\beta_1\alpha_1 rf$ is the identity on I^3 keeping E fixed. Let $g = \delta\gamma\beta\alpha$. Thus rf is isotopic to the identity on I^3 by g and f is isotopic to $r^{-1} = r$ by rg_t.

A surface (S, G, J) is said to be in *regular position* if G is in laundry position and S is partitioned into strips of type L, U, or R whose end arcs project homeomophically into R^2. A *model* corresponding to (S, G, J) is a triple (sG, tw, cr) where sG is the subdivision of $G, tw : Q \to \{-1/2, 0, 1/2\}$ is the twist function giving the twist of each strip, and cr is the crossing function that assigns to each overlapping pair (i, j) the crossing sign, $cr(i, j) \in \{-1, 1\}$, of the oriented chords E_i and E_j when projected into R^2.

Lemma 2. Every model corresponds to a uniquely embedded surface in regular position.

Proof. Suppose $M = (sG, tw, cr)$ is a model. An embedding of G in laundry position with a rectangular projection into R^2 is obtained by letting the projected chord P_i be the polygonal line with vertices: $(a_i, 0)$, $(a_i, n-1)$, $(b_i, i-n-1)$, $(b_i, 0)$ and crossings given by cr. Construct a surface (S, G, J) in regular position with twists given by sG and tw. Suppose that M also corresponds to another surface (R, H, K) in regular position. The projection of a circle-with-chords in laundry position results in a graph that has a unique embedding in R^2 [8]. There is an isotopy of R^2 carrying the projection of (H, K) to the projection of (G, J) preserving the orientation of the arcs K and J. This isotopy extends to an isotopy h of R^3 that preserves vertical lines at each stage of the isotopy. The isotopy h can be adjusted vertically to carry (H, K) to (G, J) because both have the same crossings and adjusted to carry R to S because both are defined by the same subdivision and twist function.

A homeomorphism of a solid torus is isotopic to a Dehn twist. It follows that a homeomorphism of a cylinder that is either fixed on the ends or rotates one component of the ends through an angle pi relative to the other component is isotopic to a homeomorphism that creates a strip having multiple twists of the same type.

Lemma 3. Every laundry surface (S, G, J) is equivalent to a surface (R, H, K) in regular position.

Proof. Let h be an isotopy carrying (G, J) to (H, K) in laundry position preserving the orientation of the arcs and let $S_1 = h_1(S)$. There is an $\varepsilon > 0$ and an isotopy α fixed on H such that α adjusts an ε-neighborhood in S_1 of each vertex of H so that it projects homeomorphically into R^2. There is a regular neighborhood W of H in R^3 of diameter less than ε. By the regular neighborhood theorem for pairs [6, p. 53], there is an isotopy β of R^3 that carries S_1 onto $S_2 = W \cap S_1$ and is fixed on H. Let W^3 be the inner cylinder of the cube I^3 and F the strip in I^3 of type U. The handlebody W [9] is the union of 3-cells W_q, that intersect only on their boundaries and such that there is a homeomorphism $f_q{:}W_q W_q$ such that $f_q(\partial D_1 \times I) = W_q \cap \partial W$, $f_q(A) \subset H$, $f_q(F)$ projects homeomorphically into R^2, and $f_q(F \cap E_2) \subset S_2$, for each $q \in Q$. Also, there is a homeomorphism $h_q : W_q \to W_q$ such that $h_q(F) = S_2 \cap W_q$, for each $q \in Q$. Each homeomorphism $f_q^{-1} h_q$ is isotopic to a homeomorphism that creates a strip having multiple twists of the same type. There is an isotopy γ of R^3 such that $\gamma(H) = H$ and $(\gamma(S_2), H, K)$ is in regular position. Let $R = \gamma(S_2)$. The required isotopy is $g = \gamma\beta\alpha h$.

The strips having the three twist types are superimposed in the cube. Lemma 4 shows that this allows corresponding strips of one type to be replaced with those of a second type while preserving ambient isotopy. This is simply the analogue of the result that changing a crossing in a knot or link diagram is well-defined.

Lemma 4. Suppose (S', G', J') and (S, G, J) are in regular position. Suppose $h : (S', G', J') \cong (S, G, J)$ is an ambient isotopy that carries a strip T' in S' to a strip T in S with the same twist type. The strips T' and T can be replaced with strips having a second twist type so that the resulting surfaces (R', G', J') and (R, G, J) are ambient isotopic by an isotopy that agrees with h on $cl(S' - T')$.

Proof. Let I^3 denote the cube containing the strip F' with the first twist type and A the center arc. There exist orientation preserving homeomorphisms f' and f of I^3 into R^3 carrying F' to T' and T, respectively, given by the handlebodies for S' and S. Let $U = f(I^3)$ and $U' = h_1 f'(I^3)$. It follows from the relative regular neighborhood theorem [5, p. 21], that there is an ambient isotopy α moving U' to U keeping S fixed. The map $h' = f^1 \alpha_1 h_1 f'$ is an orientation preserving homeomorphism of the cube onto itself, $h'(F') = F'$, and $h'(A) = A$. By lemma 1, h' is isotopic to a rotation r in R_π, by an isotopy β that leaves $F \cap \partial I^3$ and $A \cap \partial I^3$ setwise fixed. Because of the 3-cell structure of the handlebody for S, there is a bicollar B^3 of ∂U such that $B^3 \cap S$ is a bicollar of $\partial U \cap S$ in S. The isotopy of U onto itself, $\gamma_t = f\beta_t f^{-1}$, extends to an isotopy of R^3 with support in B^3. Let $g = \gamma\alpha h$. Suppose F is the strip in I^3 of the second type. Since $r(F) = F, g_1(f'(F')) = fr(F) = f(F)$. The set T and the regular neighborhoods U' and U can be replaced by slightly smaller sets so that g agrees with h on $cl(S' - T')$.

Suppose e is an edge of G that is subdivided into edges $x_1, \dots, x_k$. The sequence of twists at e is $s(e) = tw(x_1) \dots tw(x_k)$. Suppose tw_1 and tw_2 are two twist functions defined on subdivisions sG_1 and sG_2 of the same graph G. The *sum* $sG = sG_1 + sG_2$ and $tw = tw_1 + tw_2$ are defined by juxtaposition. That is, if $s_1(e)$ and $s_2(e)$ are the sequences of twists at an edge e of G then $s(e) = s_1(e)s_2(e)$ is the sequence of twists for the sum. Two models are said to be *equivalent* if their corresponding surfaces are equivalent.

Lemma 5. If $(sG, tw, cr) \cong (sG', tw', cr')$ then $(sG + sG'', tw + tw'', cr) \cong (sG' + sG'', tw' + tw'', cr')$ for any subdivision sG'' of G and twist function tw''.

Proof. Let (S, G, J) and (S', G', J') be corresponding surfaces in regular position and $h : (R, H, J) \cong (R', H', J')$. Let e be an edge in G and $e' = h_1(e)$. Subdivide each of these edges near their last endpoint to correspond to sG''. Partition the surfaces creating a sequence of strips of type U on the edges. Repeat at each edge. By lemma 4, the strips can be replaced to correspond to the twist function tw''.

In addition to the interior first-edges, the chord E_0 and the arc a_0a_1 (when $n > 0$) are called *first-edges*. A model is said to have *canceled* twists if only twists of the same type occur at any edge and the twist at each first-edge of G is either zero or one-half. Lemma 6 notes that every model is equivalent to one with canceled twists.

Lemma 6. $(sG, tw, cr) \cong \text{cancel}(sG, tw, cr)$.

Proof. Let (S, G, J) be a corresponding surface in regular position. Suppose $n > 0$, and F is a strip whose arc is a first-edge for a component C of the overlap graph O of G. The portion of S corresponding to C can be rotated around the x-axis so that there is only one twist in F and it is either 0 or 1/2 depending upon whether the number of twists in F is even or odd, respectively. Repeat this procedure for each of the strips whose arc is a first-edge for a component of the overlap graph by starting at a_0 and moving along the x-axis to b_0. The twists at the first-edges E_0 and a_0a_1 (when $n > 0$) can be moved along E_0 into the last strip on J leaving these first-edges with zero twist. Each edge can be modified to have twists of only one type by an isotopy that collects and cancels its twists.

Suppose the edges of G are assigned weights by a real-valued function w. The weight of a cycle is the sum of the weights assigned to the edges in the cycle. The next proposition appears in [10] its proof provides an algorithm for computing w. In the proof of lemma 9 a twist homomorphism is defined for homology modulo two. There is difficulty in defining such a homomorphism using integer coefficients. In fact, proposition 1 is false, even for integer weights, if only the cycle basis X is used. Let Y denote the family of cycles $Y_i = E_0 \cup E_i \cup a_0a_i \cup b_ib_0, i = 0, \ldots, n$. Let Z denote the family of cycles $Z(i, j) = E_i \cup E_j \cup a_ia_j \cup b_ib_j$ for each overlapping pair (i, j).

Proposition 1. Suppose the edges of a circle-with-chords are assigned weights by a real-valued function w. Then w is determined by its values on the first-edges and the cycles in $X \cup Y \cup Z$.

Proposition 2. Suppose the edges of a circle-with-chords G are assigned weights by an integer valued function w. Suppose the weight of each first-edge is zero and the weight of each cycle in $X \cup Y$ is zero. Then $w(E_i) = w(I_i) = 0, i = 0, \ldots, n$. If, in addition, the weight of each cycle in Z is a multiple of four, then the weight of each edge in G is even.

Proof. In the proof of Proposition 1, x_i, y_i, and $w(E_i)$, are zero when the cycle weights $w(E_i) + x_i$, $w(E_i) + y_i$, and $x_i + y_i$ are zero. Also, when the value of $x = w(qb_t)$ is determined, $s_1 + s_2 = 0, s_2 + z = 0$, and $s_1 + z = s_3$. Hence, $2z = s_3$, which is a multiple of four. Thus, z is even. It follows that the weight of each edge is even.

The Gordon and Litherland form $GL : H_1(S, Z) \times H_1(S, Z) \to Z$ is represented by the linking matrix M. Suppose (S, G, J) is a surface in regular position and C is an oriented cycle in G. If $C = \sum r_i X_i$ and $R = [r_i, \ldots, r_n]$ then $GL(C, C) = RMR^T$. Define the *self-linking* $self(C) = (1/2)GL(C,C)$, $twist(C)$ be the sum of the twists in C, and $writhe(C)$ be the sum of the crossing signs of C with itself. A regular neighborhood N of C in S is a Mobius band or an annulus. Orient the boundary components of N to agree with the orientation of C. It can be shown that $twist(C)$ is one-fourth of the sum of the crossing signs of ∂N with C at twists in C, $writhe(C)$ is one-fourth of the sum of the crossing signs of ∂N with C at the crossings of C, and $self(C)$ is one-fourth the total. This yields the familiar formula $self(C) = twist(C) + writhe(C)$.

Lemma 7. Suppose two surfaces in regular position have the same graph and linking matrix. Suppose first-edges have the same twist sum and corresponding crossings have the same sign. Then each edge has the same twist sum.

Proof. Define the weight w of an edge to be the sum of the twists at the edge. First-edges have the same weight. If $C \in X \cup Y$ then $writhe(C) = 0$. So $w(C) = self(C)$. If $C \in Z$ and is defined by the overlapping pair (r, s), then $w(C) = self(C) - writhe(C) = self(C) + cr(r, s)$. The cycles in $X \cup Y \cup Z$ have the same weight. Proposition 1 implies that each edge has the same weight.

A consequence of lemma 7 is that the surfaces would have the same model and hence be equivalent by lemma 2. The difficulty is that the crossings are not given by the hypothesis of theorem 1. Instead of the crossings, the proof focuses on the sidelinking numbers defined next. Suppose S is orientable and (i, j) is an overlapping pair. The point b_i has a triod neighborhood in G. Define the *positive side of* S *at* b_i using the right-hand rule and the angle at the point b_i from the chord E_i to the edge of G exiting b_i and missing the edge entering b_i. Define the *side-linking* number to be $L(i, j) = lk(X_i^+, X_j)$ where X_i^+ is the curve X_i pushed slightly toward the positive side of S at b_i. Lemma 8 gives a relation between side-linking, twists, intersection, and crossing.

Lemma 8. Suppose (S, G, J) is a surface in regular position. Suppose (i, j) is an overlapping pair. Let $t(i, j)$ be twice the sum of the twists at the arc $a_i b_j$ in J. Let $I(i, j)$ to be 1 if $t(i, j)$ is even and 0 if $t(i, j)$ is odd. Then $2L(i, j) = t(i, j) + I(i, j) + cr(i, j)$.

Proof. Each term can be expressed as a sum of crossing signs of X_i^+ and X_j. If X_i intersects X_j in S then there is a plus one crossing near b_i.

Lemma 9. Suppose (S, G, J) and (S', G', J') are surfaces in regular position with the same graph, linking matrix, and turns. Suppose that S is orientable. Then

S' is orientable and corresponding overlapping pairs have the same side-linking numbers.

Proof. For each edge e in G, let $ntw(e) = 2tw(e)$ mod 2. That is, the number of half-twists (an integer) mod 2. For any cycle C in G, let $ntw(C)$ be the sum over the edges in C. Using addition of cycles mod 2, $ntw(C + C') = ntw(C) + ntw(C') - 2ntw(C \cap C') = ntw(C) + ntw(C')$. Thus $ntw : H_1(G, Z/2Z) \to Z/2Z$ is a homomorphism. The surface S is orientable iff $ntw(C) = 0$, for all cycles C. Or, equivalently, $ntw(X_i) = 0$, for all basic cycles $X_i \in X$. Using values mod 2, $GL(X_i, X_i) = 2self(X_i, X_i) = 2twist(X_i) = ntw(X_i)$. So the surface S is orientable iff the diagonal of the linking matrix consists of even integers. Therefore, S' is also orientable. Suppose (r, s) is an overlapping pair and $L(r, s)$ and $L'(r, s)$ are the side-linking numbers for (S, G) and (S', G'), respectively. Let H and H' be the subgraphs of G and G', respectively, obtained by removing all of the chords except E_0, E_r, and E_s. Let R and R' be regular neighborhoods of H and H' in S and S', respectively. If the crossing signs, $cr(r, s)$ and $cr'(r, s)$ for G and G' differ then there is an isotopy α that rotates E_s around the x-axis so that they agree. The isotopy α can be chosen so that $\alpha(R', H')$ is in regular position. There is an isotopy β, again canceling twists, so that $(R'', H'') = \beta\alpha(R', H')$ has the same twist sum on first-edges as (R, H). By lemma 8, they have the same twist on the arc $a_s b_r$. By lemma 7, (R, H) and (R'', H'') have the same side-linking. The isotopies α and β don't change the positive side at b or the order of r and s, so $L(r, s) = L'(r, s)$.

The surface (S, G, J) is said to have *trivial linking* if (1) $tw(e) = 0$, for each first-edge e of G, (2) $self(C) = 0$, for each cycle in $X \cup Y$, (3) $self(C) \in \{-1, 1\}$, for each cycle $C \in Z$, and (4) the side-linking number $L(I, j)$ is 0 or 1 for each overlapping pair (i, j).

Lemma 10. Suppose (S, G) has trivial linking. Then $tw(E_i) = tw(I_i) = 0$, for $i = 0, \dots, n$, the twist at each edge in G is an integer, and S is orientable.

Proof. Define the weight of each edge in G to be twice the twist at the edge and apply proposition 2. Thus $tw(E_i) = tw(I_i) = 0$, for $i = 0, \dots, n$. For any cycle C in Z the self-linking and writhe are both elements of the set $\{-1, 1\}$. So its twist, which is their difference, must be even. The weight of C is a multiple of four. The twist at each edge is an integer and S orientable.

Two equivalent graphs will have equivalent regular neighborhoods. The proof of the main theorem proceeds by showing that an equivalence on these handlebodies can be extended to a carefully chosen family of disks that span the handles under the condition of trivial linking. Suppose that (S, G) has trivial linking. Let F' be the 2-cell in S consisting of the union of the strips for the edges in the arc of G. There is a regular neighborhood of G that is a handlebody W such that $S \cap \partial W = \partial S$. Let W^3 be the cylinder in the cube, F the strip of type U, and A the center arc. There is a 3-cell $C \subset W$ and a homeomorphism $w : W^3 \to C$ such that $w(A)$ is the arc of G and $w(F) = F'$.

Since the twist at each edge in the arc of G is an integer, there is a component of $F \cap (\partial D_1 \times I)$ that contains $w^{-1}(E_i \cap \partial C), i = 1, \dots, n$. There is a level preserving identification map f of $I^2 = I \times I$ onto $\partial D_1 \times I$ so that $g = wf$ identifies only points $(-1, y)$ and $(1, y)$, for $y \in I$, $g^{-1}(\partial F') = \{-1, 0, 1\} \times I$, and $g^{-1}(E_i) \subset \{0\} \times I, i = 1, \dots, n$. For $i = 1, \dots, n$, each chord E_i intersects ∂C in two points near its endpoints, a_i and b_i. To ease the notation, let a_i and b_i; also denote the inverse images of these points in $g^{-1}(E_i)$ so that $\{0\} \times I$ has the same interval structure as the arc in G. The components of $cl(W - C)$ are handles of W with cores $A_i = cl(E_i \cap (W - C))$ for $i = 1, \dots, n$ and $A_0 = E_0$. Let $-1 < x_n < \cdots < x_1 < 0$ be any sequence of points.

Lemma 11. Suppose that (S, G, J) has trivial linking. Then there is a set $\{K_0, \dots, K_n\}$ of arcs in ∂C such that K_i joins the points in $E_i \cap \partial C$ and a set $\{B_0, \dots, B_n\}$ of pairwise disjoint disks such that $\partial B_i = A_i \cup K_i$ and $C \cap \text{int } B_i = \emptyset$, for $i = 0, \dots, n$.

Proof. Suppose G has the rectangular projection as in lemma 2. It has the property that chords with larger subscripts are nearer the x-axis. The proof is by induction on n. Suppose $n = 0$ or 1. There is an ambient isotopy that untwists each edge. The required disks are immediate. Suppose that $n > 1$. There is an ambient isotopy α that untwists the chord E_n and the arc I_n leaving the remaining twists in S unchanged. There is a horizontal disk D with $\partial D = X_n$. Suppose the first point preceding b_n is not a_n then the first point preceding b_n is a right endpoint b_r for some $r < n$. In untwisting the arc, the chord E_r which overlaps E_n may wrap around the x-axis. Since $r < n$ and the side-linking number $L(r, n)$ is 0 or 1, $E_r \cap D$ can be at most one point. The positive side of S at b_n is up. The cycle X_r is pushed up. There are six possibilities. The chord E_r can pass either over or under E_n and possibly either over or under and around the x-axis. The case under-up-and-around results in $L(r, n) = 2$ and the case over-down-and-around results in $L(r, n) = -1$. Neither occur. The cases where E_r simply passes over E_n or passes under-under-and around result in $L(r, n) = 0$. The right feet of handles at E_n and E_r will pass each other when rotated clockwise. The cases where E_r simply passes under E_n or passes over-over-and-around result in $L(r, n) = 1$. The right feet of handles at E_n and E_r pass each other when rotated counter clockwise. Sliding the arc A_n across the disk $B = \alpha^{-1}(D)$ defines a handle slide so that A_n is not overlapped by A_r. There is an homeomorphism of $I \times I$ onto itself that transposes the points b_n and b_r on $\{0\} \times I$ by a corresponding rotation. Continuing, define a composition of handle slides s_n that corresponds to a product of transpositions t_n. So that starting with the arc A_n of the innermost chord E_n the handle slides in s_n move the right edge of A_n and shrink an edge of A_n until A_n no longer has any crossings. The last transposition may be assumed to also translate the two feet out of the way so that $t_n(a_n)$ and $t_n(b_n)$ are in $\{x_n\} \times I$. Define by induction a sequence of handle slides $s_n, \ldots, s_1$ and corresponding sequences $t_n, \ldots, t_1$ of transpositions. The composition of the rotations performing the transpositions defines a homeomorphism h of $I \times I$ onto itself. The arcs $h(a_i)h(b_i) \subset \{x_n\} \times I$, for $i = 1, \ldots, n$, will be pairwise disjoint. The required arcs are $K_i = gh^{-1}(h(a_i)h(b_i))$, for $i = 1, \ldots, n$,. Let K_0 be the union of $g(\{-1\} \times I)$ and two radii in the ends of the 3-cell C. There is a family of pairwise disjoint disks $\{F_0, \ldots, F_n\}$ such that $\partial F_i = g(h(a_i)h(b_i)) \cup s_i(A_i)$, for $i = 1, \ldots, n$. Reverse the sequences of slides and let B_i be the result of moving F_i, for $i = 1, \ldots, n$. These are the required spanning disks.

The reader should note in the above proof that the rotation direction was determined by the side-linking number.

Proof of Theorem 1. It is immediate that equivalent surfaces must have the same graph, linking matrix, and turns. To show the converse it suffices, by Lemma 3, to consider surfaces in regular position. Suppose (S, G, J) and (S', G', J') are surfaces in regular position with the same graph, linking matrix, and turns. There is a homeomorphism $h : (G, J) \to (G', J')$ that preserves the orientations of the arcs J and J'. Let $M = (sG, tw, cr)$ and $M' = (sG', tw', cr')$ be models corresponding to these surfaces. Let tw^{-1} denote the inverse of tw obtained by changing twists of type R to type L and those of type L to type R. Let $M_1 = cancel(sG + sG, tw + tw^{-1}, cr)$. Let $M_2 = cancel(sG' + sG, tw' + tw^{-1}, cr')$. If it can shown that $M_1 \cong M_2$, it will follow from lemmas 5 and 6 that $M \cong M'$. By lemma 2 there are surfaces (R, G, J) and (R', G', J') in regular position corresponding to M_1 and M_2. Note that (R, G) has no twists. Also note that (R, G, J) and (R', G', J') have the same turns and linking matrix. That is, the turn at an interior first-edge, which is the twist (mod-two), will still agree after the additions, as will the linking matrix, and, by lemma 6, canceling twists can be done by an isotopy. Consider the crossing relation, lemma 7, for an overlapping pair (i, j). Since R has no twists, the sum $t(i, j)$ is zero and $d(i, j) = 1$. Hence $2L(i, j) = 1 + cr(i, j)$. So the side linking number $L(i, j)$ is 0 or 1 for each overlapping pair. Thus R has trivial linking and is orientable. By lemma 9, R' is orientable and corresponding overlapping pairs have the same side-linking numbers. Since (R', G', J') has canceled twists, the twist at interior first-edges is zero or one-half. These twists must be zero because R' has the same turns as R. Thus the twist at all first-edges of G' is zero. So (R', G', J') also has trivial linking. As noted in the remark prior to lemma 11, there are 3-cells C and C' in the handlebodies W and W' for R and R' and homeomorphisms w and w' of the cylinder W^3 onto C and C', respectively. The homeomorphisms w and w' can be chosen so that $h = w'w^{-1}$ is orientation preserving and extends h on G. The twist is zero at each chord in both R and R', so h can be extended to the handles, yielding $h : (W, R, G, J) \to (W', R', G', J')$. There is an identification map f of I^2 onto $\partial D_1 \times I$ that defines maps $g = wf$ and $g' = w'f$ of I^2 into ∂C and $\partial C'$ for R and R', respectively. Since G and

G' have the same intervals, they have the same sequence of transpositions. Since R and R' have the same side-linking numbers, the transpositions have the same rotation directions. Thus the same family of arcs $\{K_i, i = 1, \ldots, n\}$ in I^2 may be used in defining spanning disks. Also, $hg(K_i) = g'(K_i)$. There is a set of spanning disks for the handles of W and W' by lemma 11. The homeomorphism h can be extended to map each slightly thickened spanning disk to the corresponding one. The homeomorphism h can be extended to S^3. Thus (R, G, J) is equivalent to (R', G', J'). This establishes that $M_1 \cong M_2$ and $M \cong M'$. That is, (S, G, J) and (S', G', J') are equivalent.

Corollary 1. Suppose N and N' are 2-manifolds with boundary (possibly empty). Suppose $\{B_1, \ldots B_r\}$ and $\{B'_1, \ldots, B'_r\}$ are families of pairwise disjoint disks in N and N' respectively. Suppose (S, G, J) and (S', G', J') are laundry surfaces such that $N - \cup \text{ int } B_i = S$ and $N' - \cup \text{ int } B'_i = S'$. If $h : (S, G, J) \cong (S', G', J')$ is an equivalence such that $h_1(\partial B_i) = \partial(B'_i)$, for $i = 1, \ldots, n$ then h_1 can be assumed to carry N to N'.

Proof. Suppose (G_1, J_1) is any graph in laundry position and W_1 is a regular neighborhood of G_1. Then $S^3 -$ int W_1 is a handlebody. To see this, consider (S_1, G_1, J_1) where S_1 has no twists. Then (S_1, G_1, J_1) has trivial linking. There is a regular neighborhood of G_1 that is a handlebody W_2 such that $S_1 \cap \partial W_2 = \partial S_1$. By lemma 11, there is a family of disks that when thickened will form handles for $S^3 - int W_2$. By the uniqueness of regular neighborhoods, $S^3 -$ int W_1 is also a handlebody. So, in proving the corollary, let $K = (\cup B'_i) \cup (\cup h_1(B_i))$ and let W be a regular neighborhood of S' mod K in S^3 [5]. Let $W' = S^3 -$ int (W). Since the disks $h_1(B_i)$ and B'_i have the same boundary in the handlebody W', h can be adjusted in this handlebody by cut-and-paste so that $h_1(B_i) = B'_i$, for $i = 1, \ldots, n$. Thus $h_1(N) = N'$.

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